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A CHARACTERIZATION OF ALTERNATING GROUPS BY THE SET OF ORDERS OF MAXIMAL ABELIAN SUBGROUPS

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Abstract: We prove that alternating groups with three prime graph components are uniquely determined by the set of orders of maximal abelian subgroups.

Keywords: abelian group, alternating group, prime graph component

It was first time under consideration in [1] how the orders of maximal abelian subgroups influence the structure of the group. The following simple groups were proved to be uniquely determined by the set of orders of maximal abelian subgroups: the k_3 groups, A_n, $n \leq 10$, Mathieu groups, Janko groups, $PSL(2, 2^n)$, and $Sz(2^{2m+1})$.

Here we will prove that the alternating groups A_p with p and $p-2$ primes can be uniquely determined from the set of orders of maximal abelian subgroups and give some subsets of the set of orders of maximal abelian subgroups which can also determine the groups.

Definitions and notations: $\Gamma(G)$ denotes the prime graph of G; $t(\Gamma(G))$ denotes the number of prime graph components of G; π_i, $1\leq i \leq t(\Gamma(G))$, denote the set of vertices of prime graph components of G. If G is of even order then π_1 always denotes the even prime graph component of G. We let $\pi_e(G)$ denote the set of orders of elements of G, and $\pi(G)$, the set of prime divisors of $|G|$. Let $M(G) = \{n = |N| \mid N$ be a maximal abelian subgroup of $G\}$. Let p be a prime; a, an integer; and $p^n \parallel a$ means $p^n \mid a$ and $p^{n+1} \nmid a$. Let m and n be positive numbers. We let (m, n) stand for the greatest common divisor of m and n.

Assume that $\pi_1, \pi_2, \dots, \pi_t$ are all prime graph components of G. Then $|G| = m_1 m_2 \dots m_t$, where $\pi(m_i) = \pi_i$, $i = 1, 2, \dots, t$. We call $m_1, m_2, \dots, m_t$ the *order components* of G (see [2]). Using the classification theorem of finite simple groups, as well as [3] and [4], we can list the order components of finite simple groups with nonconnected prime graphs in Tables 1–4 of [2].

Lemma 1. *If G is a finite group with more than one prime graph component then one of the following holds:*

(a) *G is Frobenius or 2-Frobenius and the prime graph of G has exactly two prime graph components;*

(b) *G has a normal series $H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is simple, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |\mathrm{Aut}(K/H)|$. Moreover, any odd order component of G is also an order component of K/H.*

Proof. The lemma follows from the definition of order component, together with Theorem A and Lemma 3 in [4].

By the definition of $M(G)$, the following lemma is easy:

Lemma 2. *For G and M assume $M(G) = M(M)$. Then G and M have the same prime graphs.*

Lemma 3. *For G and M assume $M(G) = M(M)$. If the prime graph of M has isolated points and the Sylow subgroups corresponding to these primes are of prime order then the set of odd order components of K/H in Lemma 1 is a subset of order components of G.*

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PROOF. By Lemmas 1 and 2, the odd components of the prime graph of K/H are some of those of M. If the condition of the lemma holds, suppose that the prime graph of M has an isolated point p and the p-Sylow subgroup of M is of order p, then the odd order components of K/H must be a power of p, which is exactly the order of a Sylow subgroup. If the order of the related Sylow subgroup K/H is not p then the latter contains an abelian subgroup of order p^2, but the maximal abelian subgroup of G with order a power of p is exactly p.

Lemma 4. *A_n with $n > 8$ has a unique faithful modular 2 representation of least degree, this degree $n-1$ or $n-2$ according as n is odd or even. These representations are realizable over $GF(2)$ (see [5]).*

Lemma 5. *If $p > 2$ then A_n with $n > 6$ has a respectively 2 and 1 faithful modular p representation of least degree, this degree $n-1$ or $n-2$ according as $p \nmid n$ or $p \mid n$. These representations are realizable over $GF(p)$ (see [6]).*

Theorem 1. *Assume that G is a finite group and M is an alternating group A_p, where p and $p-2$ are primes. If $M(G) = M(M)$ then $G \cong M$.*

PROOF. By Lemma 2 G and M have the same prime graph components. Because M has exactly three order components, we see that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is simple group with two odd order components equal to those of A_p, e.g., p and $p-2$. Since the two odd order components of A_p with p and $p-2$ primes have difference 2, by the table of the order components in [2], we infer that K/H may be one of: $A_2(4)$, ${}^2E_6(2)$, J_3, Suz and $A_{p'}$, where p' and $p'-2$ are primes.

If $K/H = A_2(4)$ then $p = 7$, $M = A_7$, and $|G/K| \mid 2$. Since $M(M) = \{4, 12, 5, 7\}$, the order of the center of Sylow subgroups of H divides 3 or 4. But these centers are normal in G, and so the result of their order minus 1 should be divided by the product of odd order components, e.g. 35; a contradiction.

If $K/H ={}^2 E_6(2)$ then $p = 19$, $M = A_{19}$. Right now A_{19} has subgroups of order 55 and 5×13, but K/H has not. Therefore, the subgroups of order 55 and 5×13 must lie in K. Hence, $5 \mid |H|$. But A_{19} has a maximal abelian subgroup of order 5^3, which means that $Z(H)$ is of order 5, 5^2, or 5^3. Obviously $Z(H)$ is normal in G, and so $17 \times 19 \mid |Z(G)| - 1$; a contradiction. Therefore, $H = 1$, and

$$^2E_6(2) \trianglelefteq G \trianglelefteq \mathrm{Aut}(^2E_6(2)).$$

Since M has an abelian subgroup of order 13×5, G must have an abelian subgroup R of the same order, e.g. 13×5. Hence, RK/K is of order 5, which implies that $5 \mid |\mathrm{Out}(K)|$; a contradiction for $|\mathrm{Out}(K)| = 6$.

If $K/H = J_3$ then $p = 19$, $M = A_{19}$. Because $5 \parallel |J_3|$ one has that $5 \mid |H|$. Since the order of the maximal abelian subgroup of A_{19} is 25, the order of the center W of the 5-Sylow subgroup of H is of order ≤ 25. But $W \trianglelefteq G$, which implies that $17 \times 19 \mid |W| - 1$; a contradiction.

If $K/H = Suz$ then $p = 13$, $M = A_{13}$. If $3 \mid |H|$ then the order of the 3-Sylow subgroup of H is of order 3^i, where $i = 1, 2, \ldots, 5$. Let S_3 be the Sylow 3-subgroup of H. Since the odd order components of G are 11 and 13; therefore, 11×13 divides $|S_3| - 1$, which is impossible. Hence, $3 \nmid |H|$. Thus all 3-subgroups are isomorphic to those in K/H. Notice that $K/H = Suz$ has an abelian subgroup of order 3^5, which implies that G has a subgroup of the same order. But $M = A_{13}$ has no subgroup of order 3^5; a contradiction.

By now, we proved that $K/H = A_{p'}$, where p' and $p'-2$ are primes. So it is easy to see that $K/H = A_p$. In case $H \neq 1$ let S_r be the Sylow subgroup of H. Then $Z(S_r) \trianglelefteq G$. Assume that K acts on $Z(S_r)$. We have that K/H acts faithfully on $Z(S_r)$ since $C_K(Z(S_r)) = H$. By Lemmas 4 and 5, this means that

$$|Z(S_r)| \geq p - 1.$$

By $M(M) = M(G)$ we have $|Z(S_r)| = p$ for $p > 7$, which implies that K/H is cyclic; a contradiction. Therefore, $H = 1$, $A_p \leq G \leq S_p$. Since A_p has an abelian subgroup of order $2(p-2)$ and A_p has not, we see that $G = A_p$. For $p = 5$ or 7, by [1], we come to $G = M$.

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